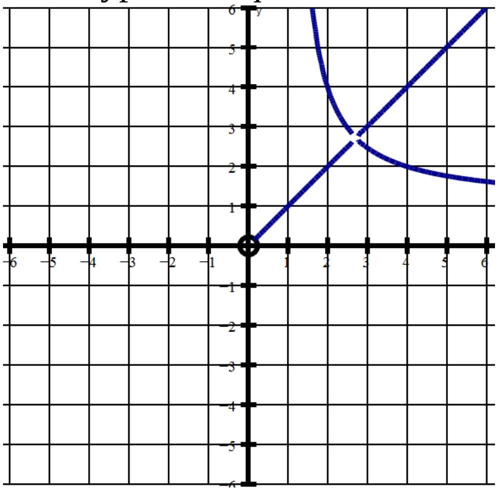
THE PERPLEXING PROBLEM OF $y^x = x^y$

If we type this equation into the Autograph program we get the following graph:



However this is not the whole story!
If we examine the part of the graph which is the same as the line $y = x$ we can see that if we let $y = x = b$ then obviously b^b **always** equals b^b

$y^x = x^y$

$1^1 = 1^1$

$2^2 = 2^2$

$3^3 = 3^3$

ALSO:

$(\frac{1}{2})^{\frac{1}{2}} = (\frac{1}{2})^{\frac{1}{2}}$

$(\frac{1}{3})^{\frac{1}{3}} = (\frac{1}{3})^{\frac{1}{3}}$

$(\frac{5}{8})^{\frac{5}{8}} = (\frac{5}{8})^{\frac{5}{8}}$

But the negative numbers also fit the equation $y^x = x^y$

Suppose $y = x = -1$ then $(-1)^{-1} = \left(\frac{1}{-1}\right)^{+1} = -1$

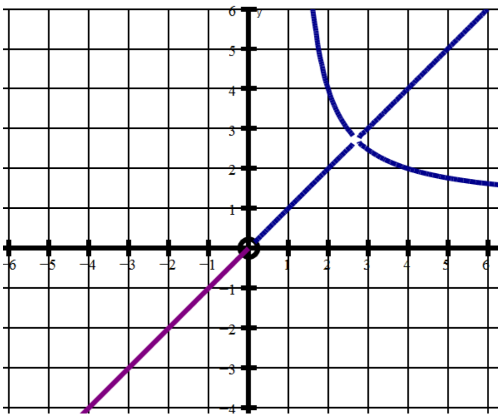
Also, if $y = x = -2$ then $(-2)^{-2} = \left(\frac{1}{-2}\right)^{+2} = \frac{+1}{4}$

Also, if $y = x = -3$ then $(-3)^{-3} = \left(\frac{1}{-3}\right)^{+3} = -\frac{1}{27}$

Also, if $y = x = -\frac{1}{2}$ then $= \left(\frac{1}{-2}\right)^{-\frac{1}{2}} = (-2)^{\frac{1}{2}} = i\sqrt{2}$

It does not matter that this is an imaginary number!
All that matters is that $y^x = x^y$ and in this case $= i\sqrt{2}$.

This means we should extend the above graph as follows: **(purple)**



The case of $y = x = 0$ is a concern of course.

Mathematicians always "shy away" from things like this with a glib comment such as "this is not defined".

I think that this is fine in some cases like $\frac{0}{0}$ which we say is "indeterminate".

I like this explanation:

If $a \times b = c \times d$ then $\frac{a}{c} = \frac{d}{b}$

Suppose $a = 6$, $b = 0$, $c = 7$ and $d = 0$

So if $\mathbf{6 \times 0 = 7 \times 0}$ then $\frac{6}{7} = \frac{0}{0}$

In other words $\frac{0}{0}$ can equal ANYTHING!

(Just substitute any numbers for a and c)

It is "indeterminate" as it is, BUT we can "determine" it.

eg $\lim\limits_{h \to 0} \frac{2xh + h^2}{h} = \frac{0}{0}$ ***if we just let $h = 0$***

BUT if we simplify it first:

$\lim\limits_{h \to 0} \frac{h(2x + h)}{h}$ ***(here we can cancel the h's because h is just a small value)***

$= \lim\limits_{h \to 0} (2x + h)$

$= 2x$

HOWEVER I think 0^0 is a bit different.

Consider $\lim\limits_{b \to 0} b^0$

Obviously $(0.000000001)^0 = 1$ so $\lim\limits_{b \to 0} b^0 \to 1$

Compare with $\lim\limits_{b \to 0} 0^b$

Obviously $0^{0.000000001} = 0$ so $\lim\limits_{b \to 0} 0^b \to 0$

I think that 0^0 can only be 0 or 1 and in this case I believe the sensible conclusion is that $0^0 \to 0$ thus completing the line $y = x$.

So instead of saying 0^0 is "NOT DEFINED" it seems sensible to simply "DEFINE" it as being equal to 0 in this particular case.

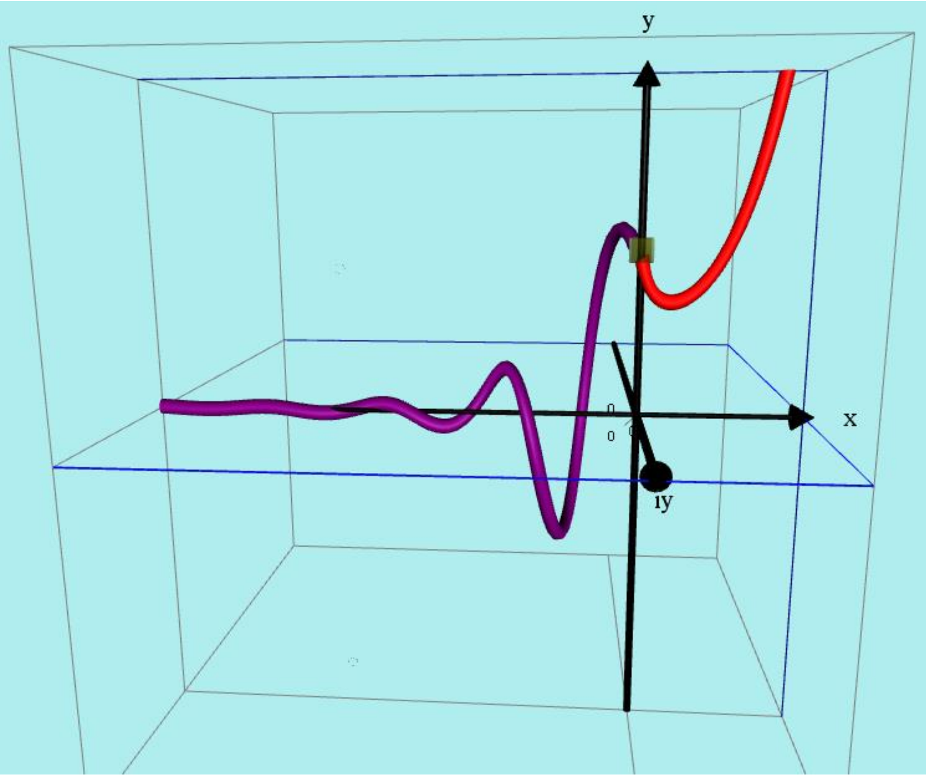
(But any purist is welcome to exclude this point if desired.)

But if $x = y = 0$ then whatever 0^0 equals (either 0 or 1) then y^x still equals x^y.

Incidentally for the graph of $y = x^x$ we have the same problem when $x = 0$ because $y = 0^0$

SEE http://screencast.com/t/m4fmGwmkrT9

The full graph of $y = x^x$ for REAL values of x (but allowing imaginary y values) is below:



Clearly $y = \lim_{x \to 0} x^x$ ***approaches*** $y = 1$ ***from*** ***<u>the left and from the right</u>.***

So instead of saying 0^0 is "NOT DEFINED" it seems sensible to simply "DEFINE IT" as being equal to 1 in this particular case.

Please view these videos…

$y = x^x$ 2020

https://www.screencast.com/t/Vkz2HWA27

$y = x^x$ extended with new insight

https://www.screencast.com/t/uo7NlUqH

The most interesting types of points on $y^x = x^y$ are those like *(2, 4) and (4, 2)* because $y^x = 4^2 = 16$ and $x^y = 2^4 = 16$

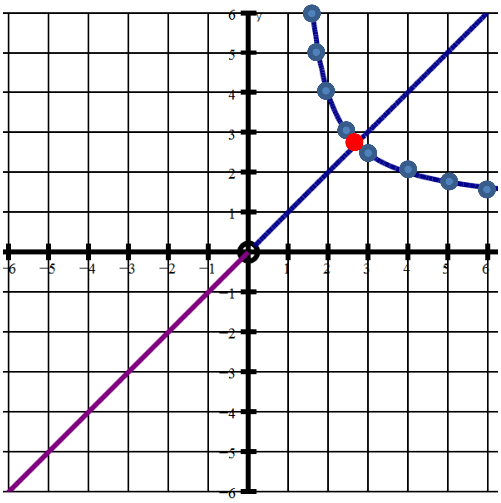
Suppose we choose $\boldsymbol{y = 5}$, then we need to solve $\boldsymbol{5^x = x^5}$ to find the $\boldsymbol{x}$ value**.**
Either solving ***graphically*** by finding the intersection of $\boldsymbol{Y = 5^x}$ and $\boldsymbol{Y = x^5}$
or using the ***equation solver*** on a graphics calculator, we get: $\boldsymbol{x = 1.764921915}$

TESTING: $\boldsymbol{5^{1.764921915} = 17.1248777}$
$\boldsymbol{\text{and } 1.764921915^5 = 17.1248777}$

Suppose we choose $\boldsymbol{y = 6}$, then solving $\boldsymbol{6^x = x^6}$ we get $\boldsymbol{x}$ = 1.624243846

TESTING: $\boldsymbol{6^{1.624243846} = 18.36146714}$
$\boldsymbol{\text{and } 1.624243846^6 = 18.36146714}$

Choosing $\boldsymbol{y}$ = 3 we get $\boldsymbol{x} \approx 2.478$ so we can plot (2.478, 3) and (3, 2.478)
Points like the above examples, produce the part of the curve which resembles a hyperbola. It is actually very close to $\boldsymbol{y - 1 = \dfrac{3}{x-1}}$



The two parts of the graph cross at
$\boldsymbol{x = 2.71828}$ and $\boldsymbol{y = 2.71828}$
which of course is x = y = e.

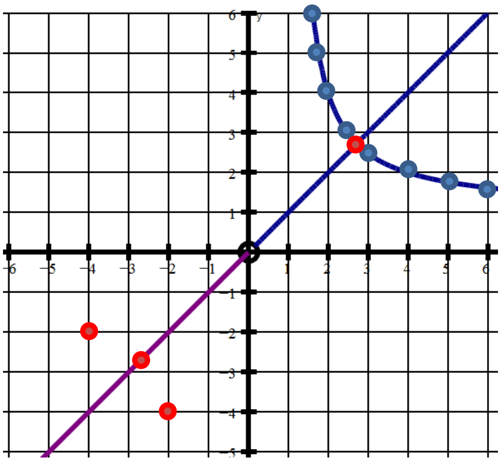
It occurred to me that I should also try some negative numbers.
I will test $\boldsymbol{x = -2, y = -4}$

TESTING: $\boldsymbol{(-2)^{-4} = (-\tfrac{1}{2})^4 = \dfrac{1}{16}}$

$\boldsymbol{(-4)^{-2} = (-\tfrac{1}{4})^2 = \dfrac{1}{16}}$

Also, considering $\boldsymbol{x = y = -2.718}$
Obviously $(-2.718)^{-2.718} = (-2.718)^{-2.718}$
The fact that $(-2.718)^{-2.718} = \boldsymbol{-0.04177 - 0.05114i}$ which is a complex number, does not matter as long as it fits: $\boldsymbol{y^x = x^y}$

So we can put the points (-2, -4) and (-4, -2) and (-2.718, -2.718) on the graph.
See below:



HOWEVER, trying *$x = -1.76492$ and $y = -5$*

$(-1.76492)^{-5} = -0.05839457758$ BUT $(-5)^{-1.76492} = 0.04318 + i\ 0.03931$

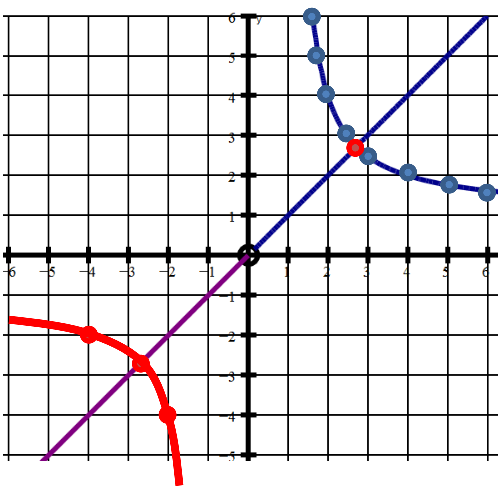
Disappointingly, this does not fit the equation $y^x = x^y$ because $y^x \neq x^y$

Similarly, trying $x = -2.478$ ***and*** $y = -3$

$(-2.478)^{-3} = -0.0657$ ***BUT*** $(-3)^{-2.478} = 0.00454 - i\ 0.0656$
Also this does not fit the equation $y^x = x^y$ because $y^x \neq x^y$

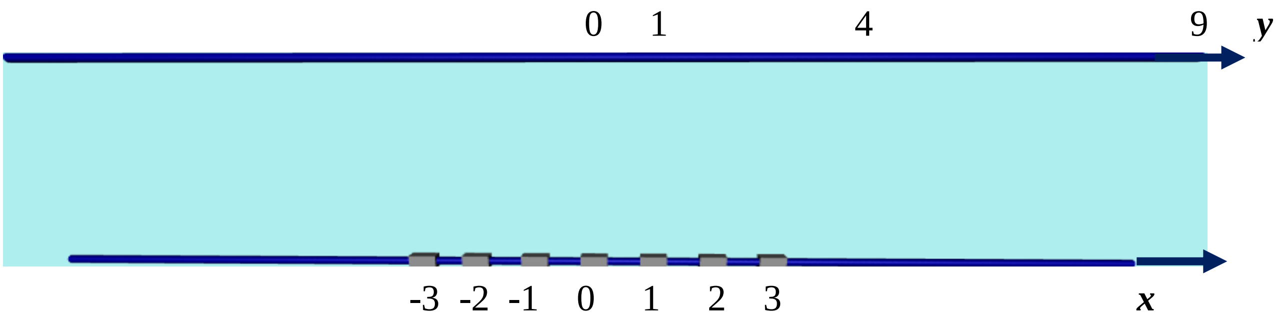
Obviously, I was <u>hoping</u> that the part of the curve resembling a hyperbola would be "reflected" or "rotated" to join up the points (-2, -4) and (-4, -2) and (-2.718, -2.718) in the 3rd quadrant.

See **RED CURVE** below:

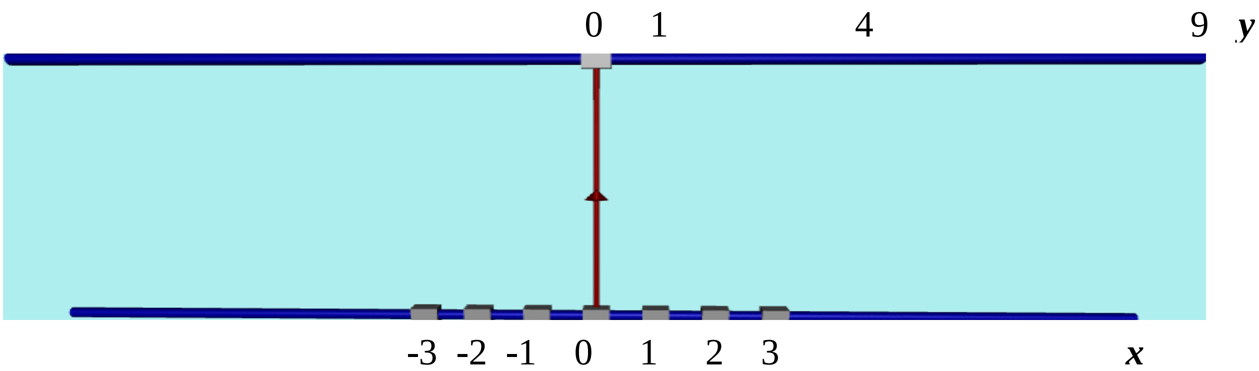


The following is a slight diversion but it does apply to this problem:

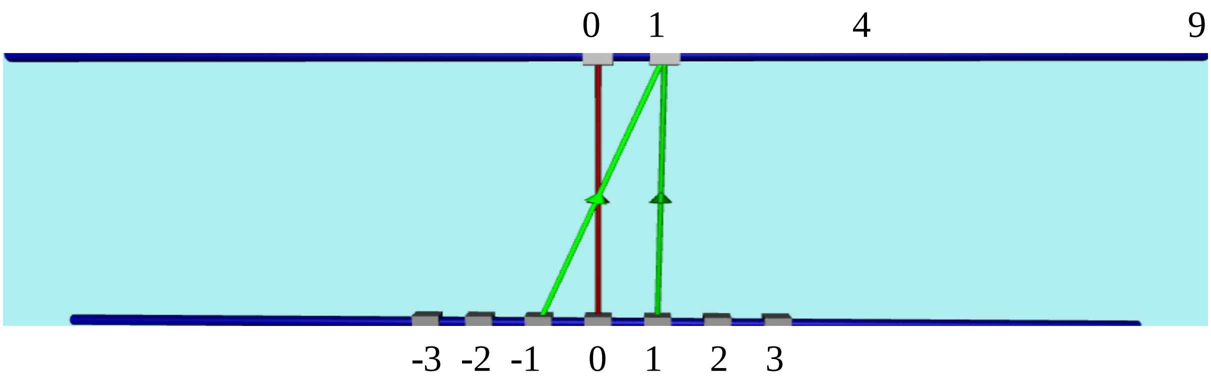
Think of $\boldsymbol{y = x^2}$ *as a process of* ***MAPPING x*** *values from an* ***x axis*** *onto* ***y*** *values* on a ***y axis*** as shown below:



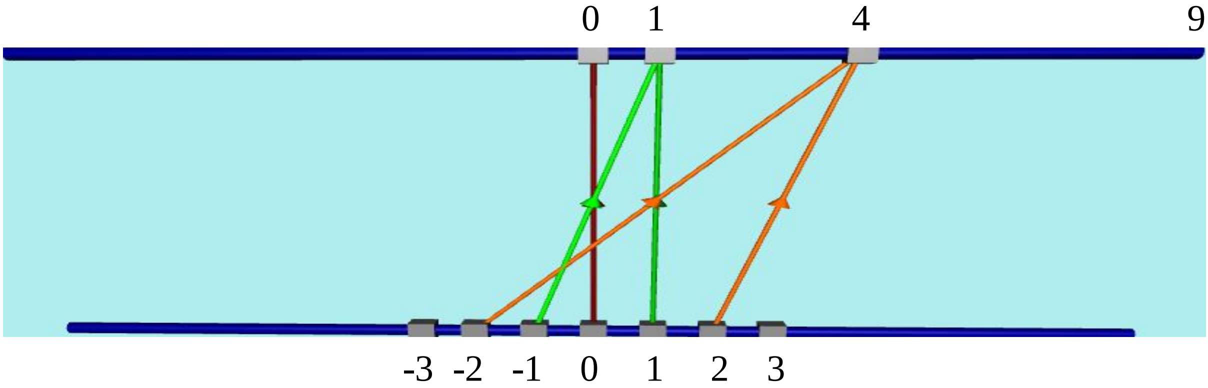
Firstly $0^2 = 0$ so we join $\boldsymbol{x} = 0$ to $\boldsymbol{y} = 0$



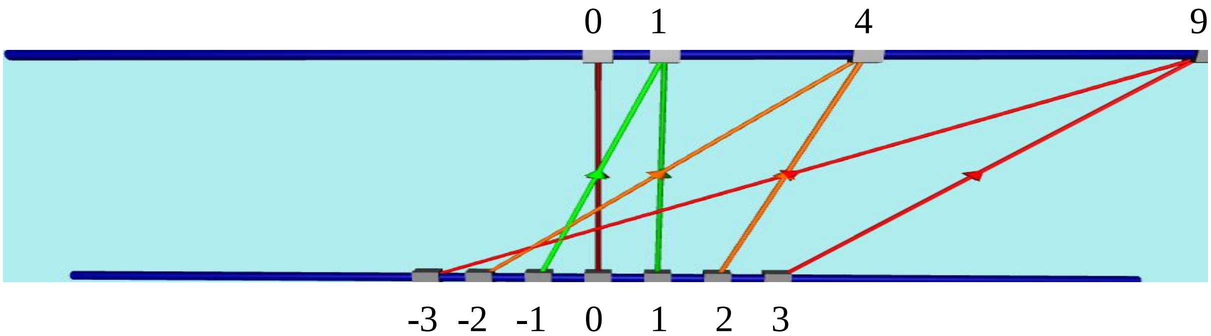
Now if $\boldsymbol{x = \pm 1, y = +1}$



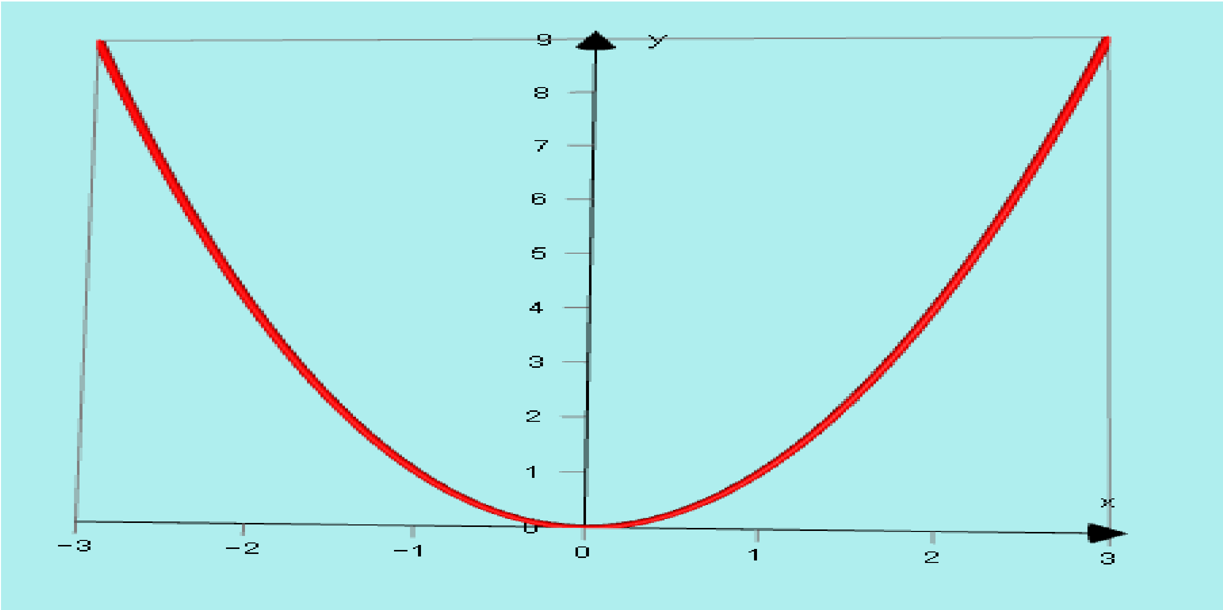
If x = ±2, y = +4



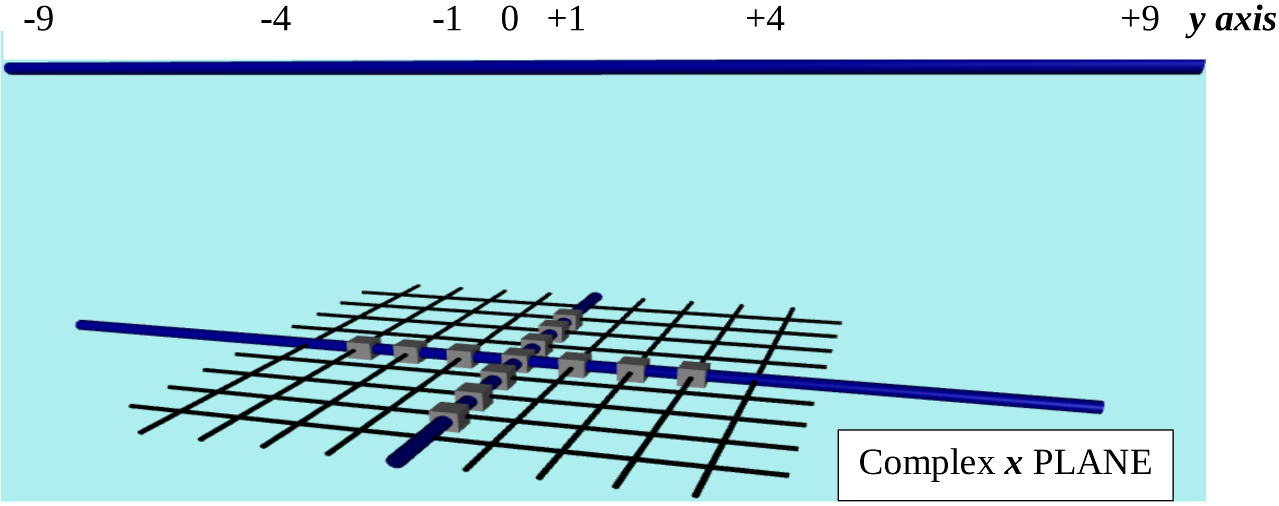
And if x = ±3, y = +9



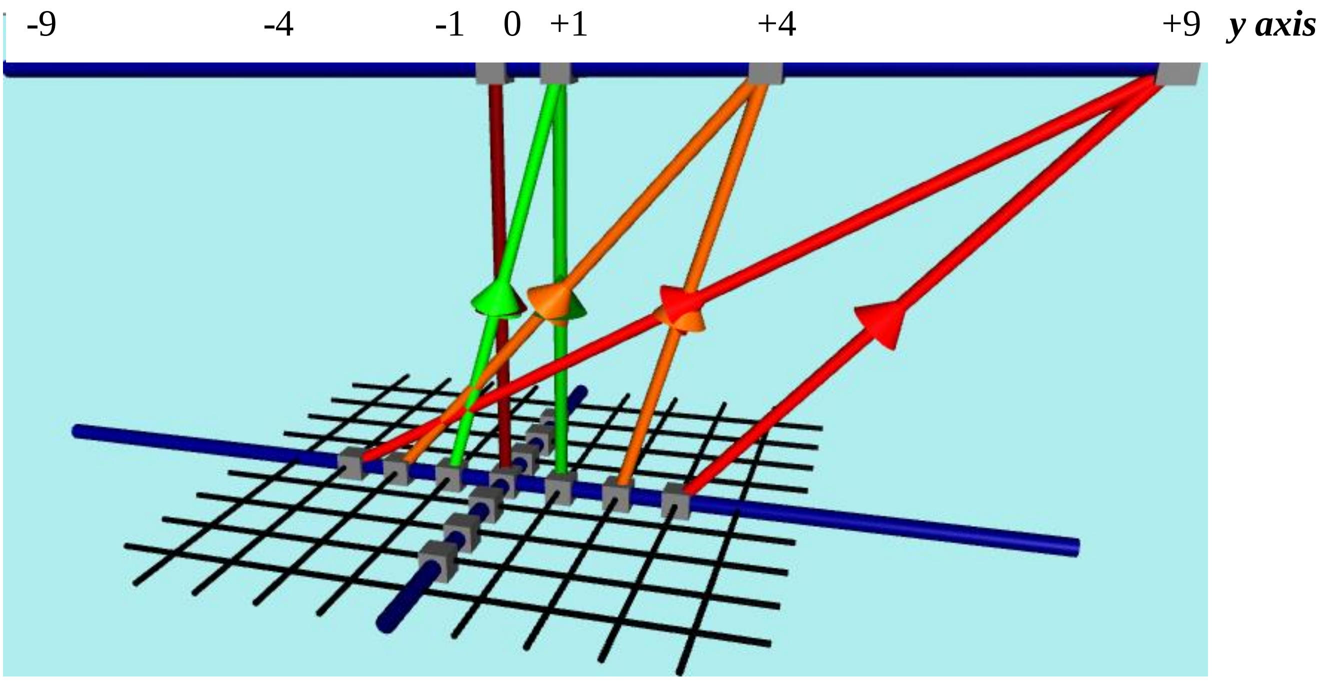
This of course produces the "normal" parabola $y = x^2$.



But now let us repeat this process of MAPPING ***x*** values from an ***x PLANE*** onto ***y*** values on a ***y axis*** as shown below:



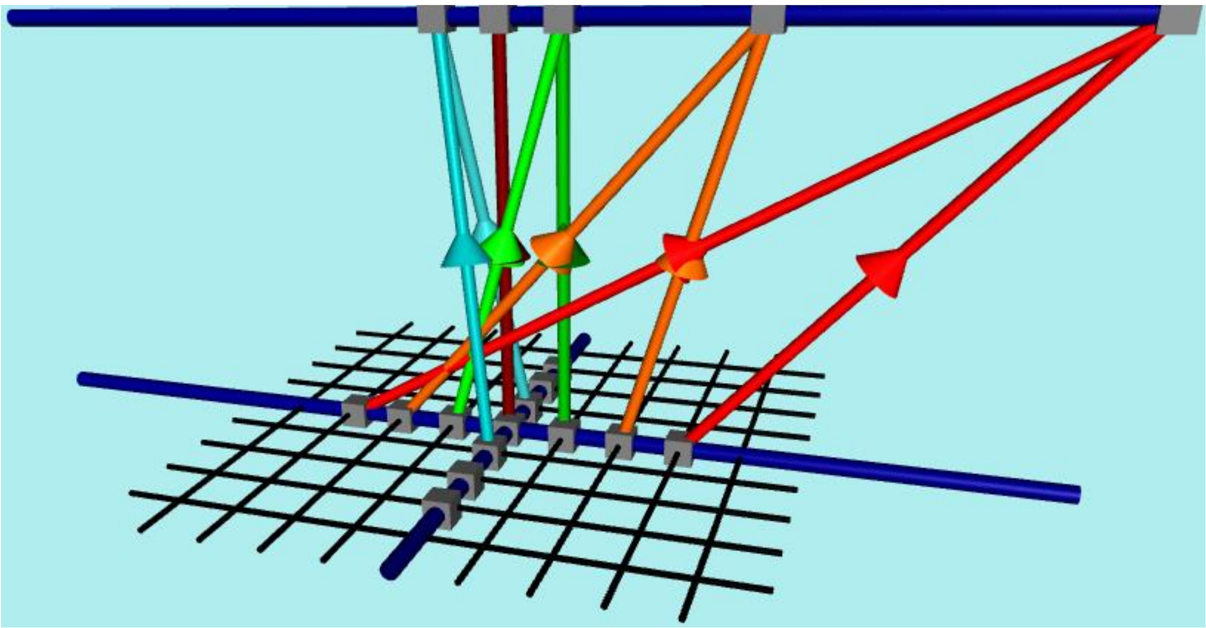
The diagram below shows ***x = ±3, y = +9 (red), x = ±2, y = +4 (orange)***
x = ±1, y = +1 (green) and x = 0, y = 0 (brown)



BUT NOW WE CAN ADD SOME IMAGINARY x VALUES WHICH PRODUCE REAL y VALUES. (This is the whole idea of Phantom Graphs!)

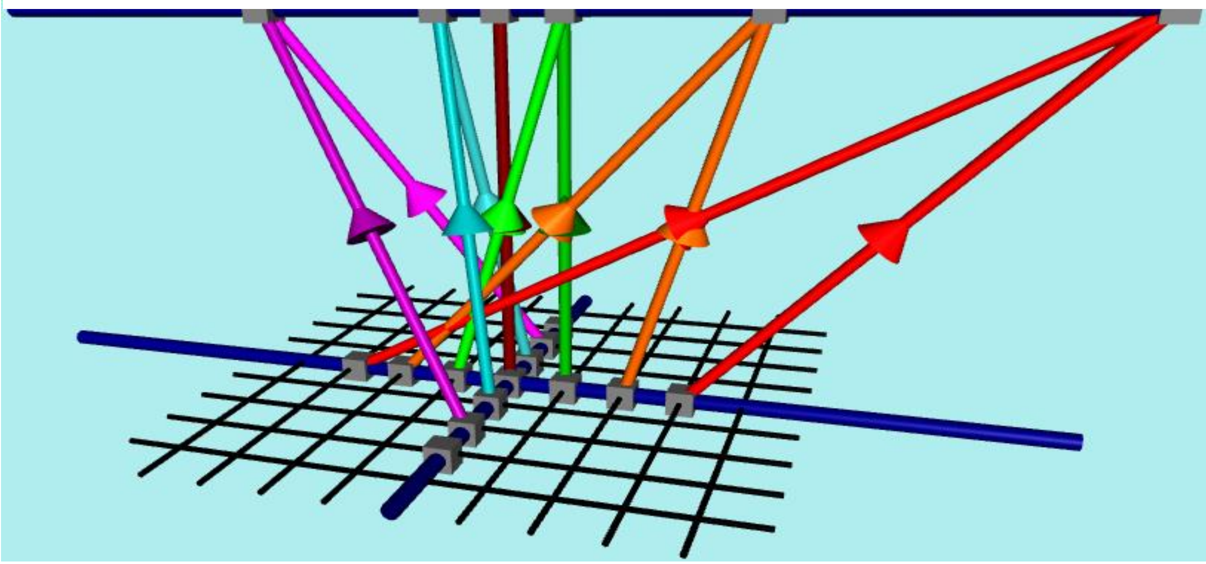
Here we add $x = \pm\, i$ which map onto $y = -\,1$ (turquoise)

-9 -4 -1 0 +1 +4 +9 ***y axis***

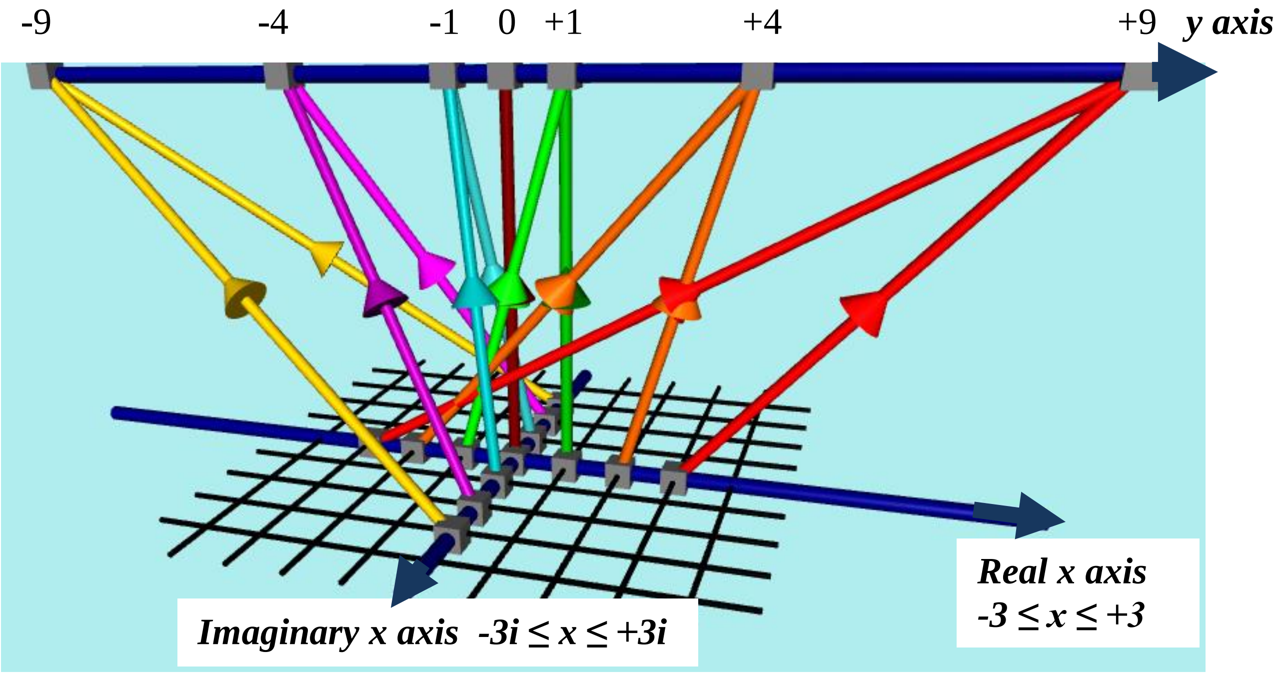


Now we add $x = \pm 2i$ and $y = -\,4$ (purple)

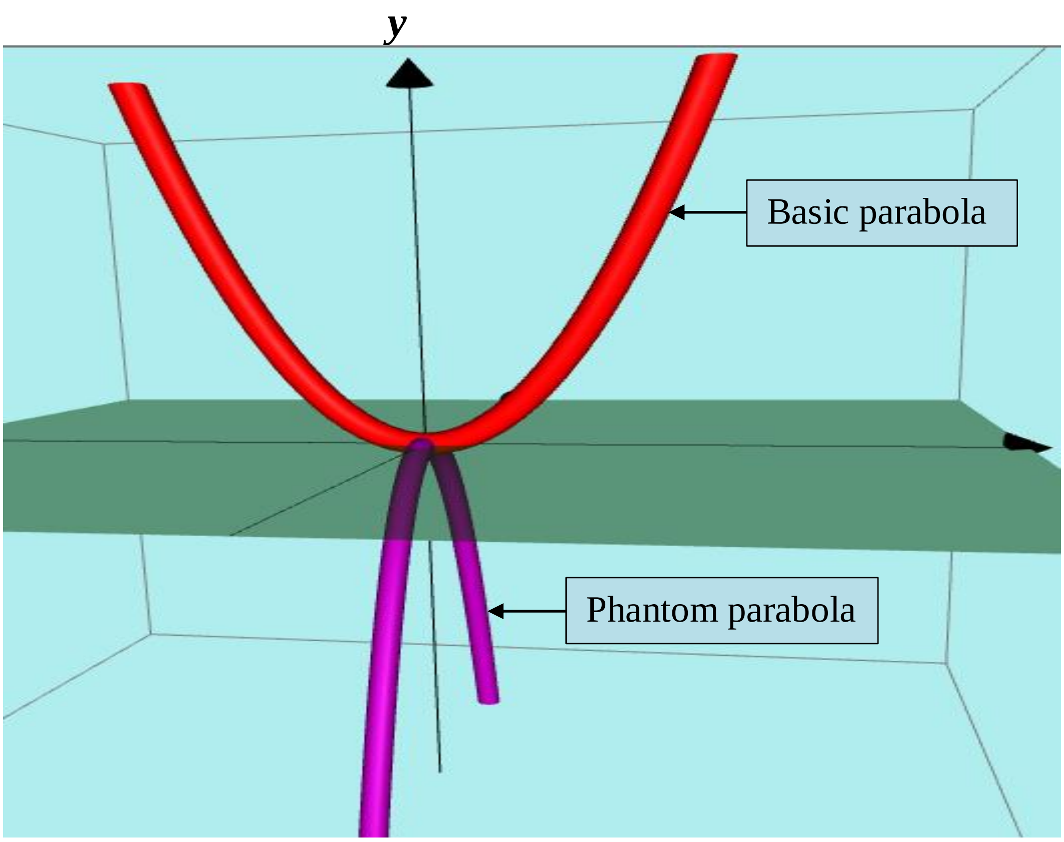
-9 -4 -1 0 +1 +4 +9 ***y axis***



And finally ***x* = ±3*i* and *y* = – 9** ***(yellow)***



This of course produces the basic PHANTOM GRAPH of y = x² if we use the complex x plane and place the vertical y axis through it.

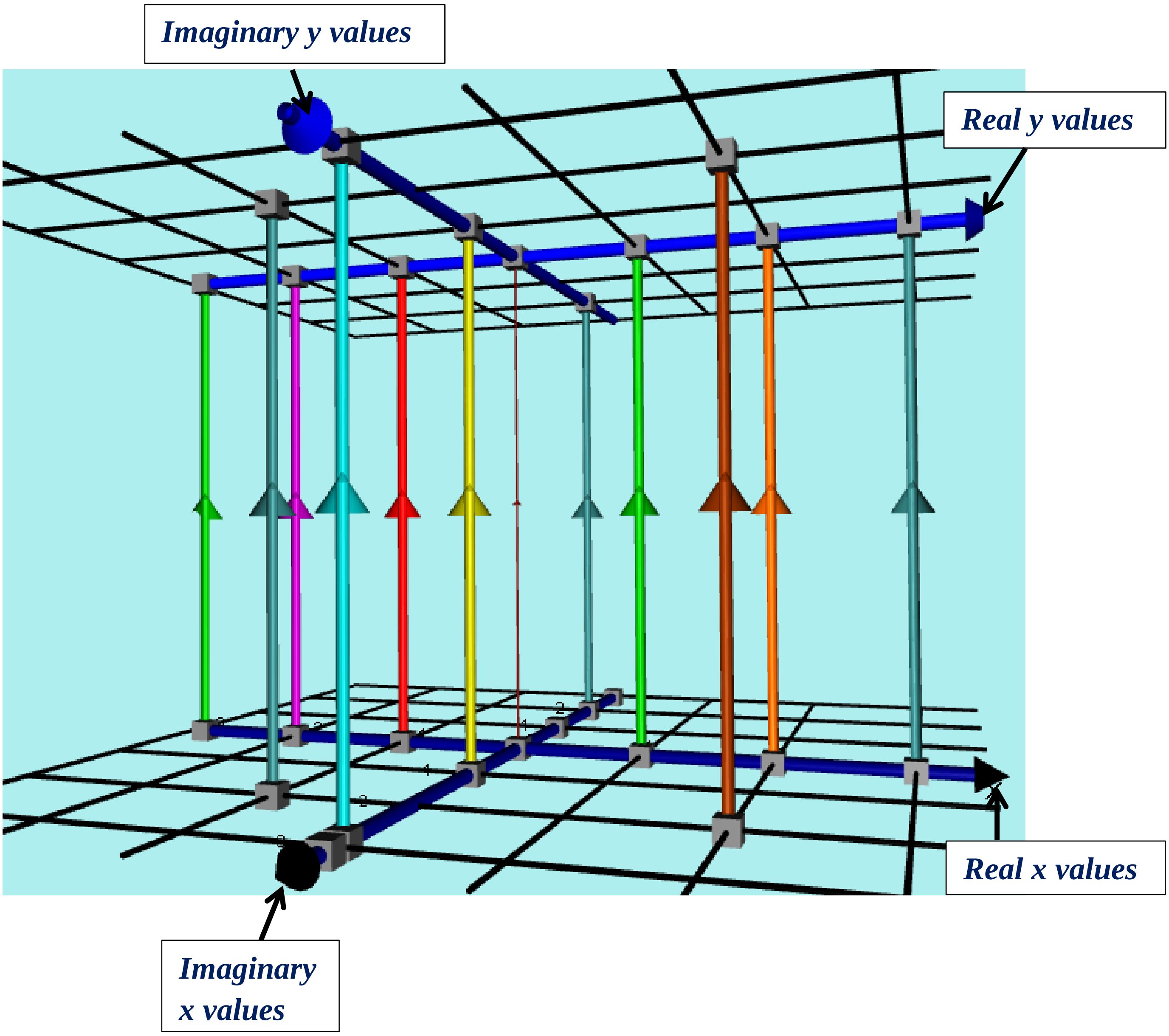


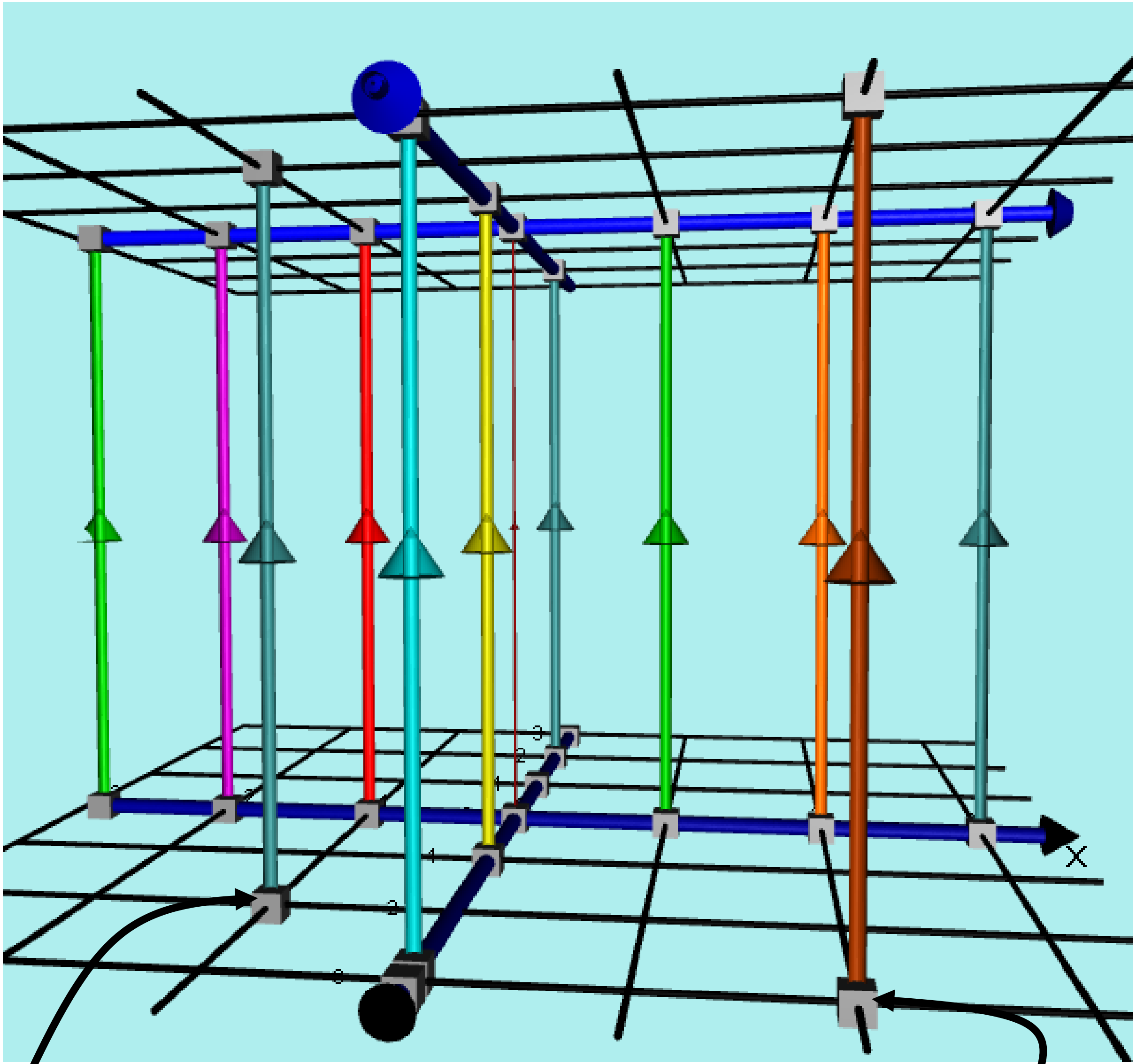
Please see video: Intro To Phantom Graphs

https://www.screencast.com/t/6Owyrraog5t

The whole point in the last 5 pages was to use the idea of mapping ***complex x values*** **onto** ***complex y values*** **for the problem $y^x = x^y$**

I established earlier that ALL real or imaginary values such as $x = y = a + ib$ must satisfy $y^x = x^y$ so the following diagram indicates this.





This is x = -1 + 2i
mapping onto y = -1 + 2i

This is x = 2 + 3i
mapping onto y = 2 + 3i

The special REAL points referred to earlier, such as the pairs:

x = 2, y = 4 and x = 4, y = 2

x = 3, y = 2.48 and x = 2.48, y = 3

x = 5, y = 1.77 and x = 1.77, y = 5

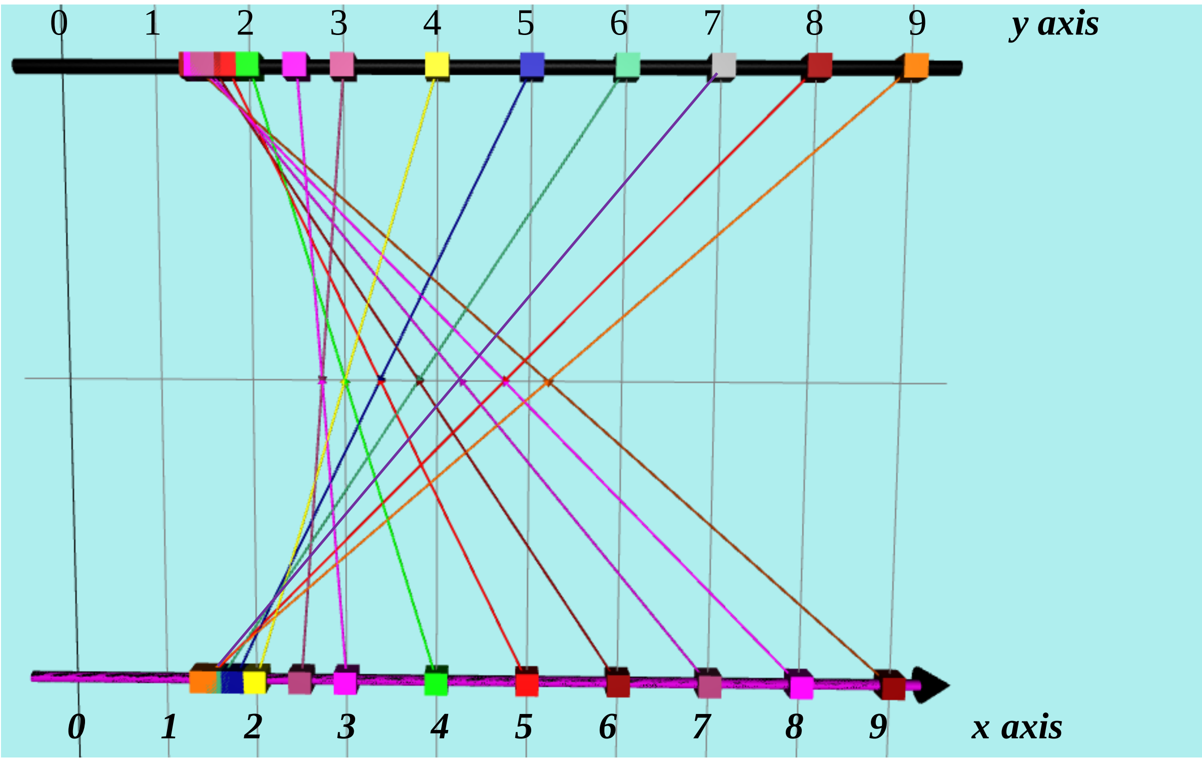
x = 6, y = 1.62 ***and x = 1.62, y = 6***

x = 7, y = 1.53 ***and x = 1.53, y = 7***

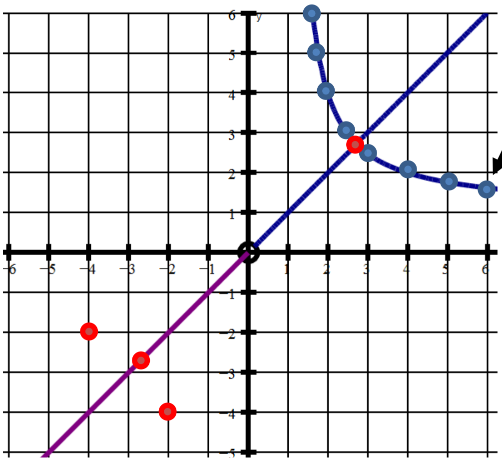
x = 8, y = 1.46 ***and x = 1.46, y = 8***

x = 9, y = 1.41 ***and x = 1.41, y = 9***

…can also be placed on a mapping of an x axis to a y axis.



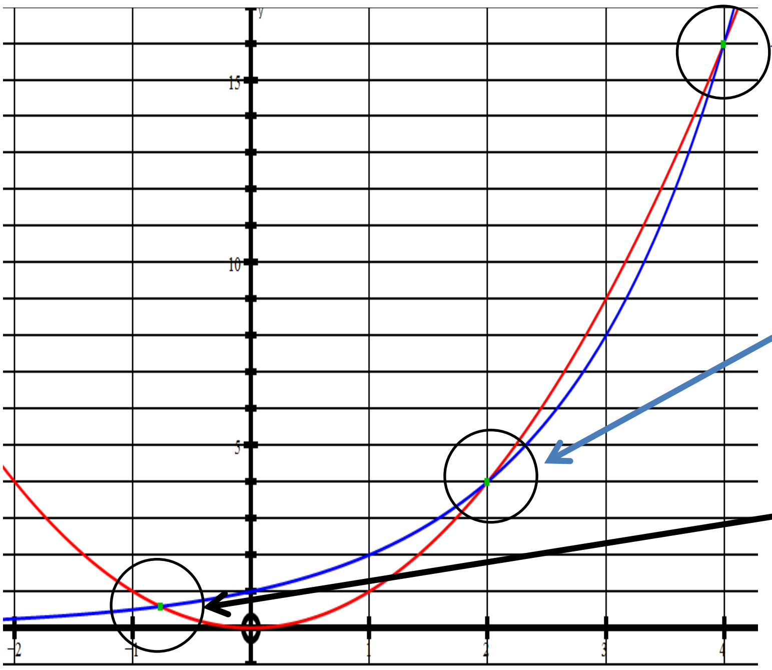
(This reminds me of the "curve stitching" that young children do!)



The above diagram represents the points which form this curved section resembling a hyperbola.

<u>SOME MORE SPECIAL *REAL* POINTS!</u>

Earlier, I referred to the "nice" whole number points (2, 4) and (4, 2) which fit the equation $y^x = x^y$.
Suppose we were not aware of these solutions and we say to ourselves,
"If y = 2, what would x be?"
ie Find x if $2^x = x^2$
If we think of this as the intersection of two graphs we could proceed as follows:
Draw $Y = 2^x$ ***and*** $Y = x^2$ ***(I am using a capital Y because these Y values are not the same as the y values in the equation!)***



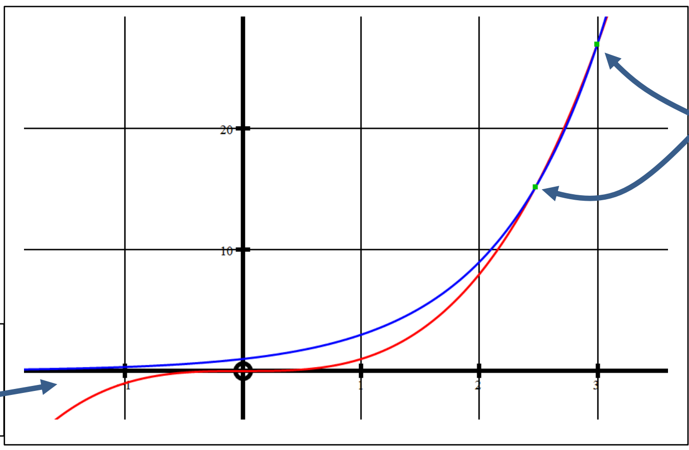
This intersection point gives the value $x = 4$ so that $2^4 = 4^2$ (both equal 16)

The obvious solution is of course $x = 2$ because $2^2 = 2^2$!

HOWEVER, there is a 3rd solution which is at $x \approx -0.7667$

If we test this 3rd solution we get $2^{(-0.7667)} = 0.5878$
and $(-0.7667)^2 = 0.5878$

This method will not produce 3 solutions for ODD y values such as $y = 3$ because the graphs $Y = 3^x$ and $Y = x^3$ only intersect TWICE.

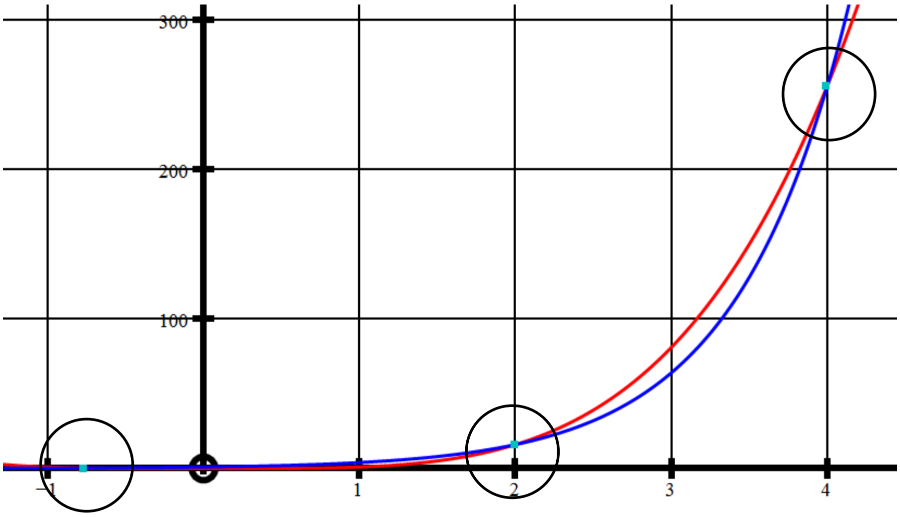


Intersection points

NO intersection occurs here.

We will only get solutions for EVEN *y* values 4^x, 6^x, 8^x …

If we draw $Y = 4^x$ and $Y = x^4$ we get graphs which intersect 3 times.



The x values at the intersection points are x = 4, 2 and – 0.7667 (again)

CHECK:

$$4^{(-0.7667)} = 0.3455$$
$$(-0.7667)^4 = 0.3455$$

We can have: x = 4, y = – 0.7667
AND x = – 0.7667, y = 4

If we draw $Y = 6^x$ and $Y = x^6$ we also get graphs which intersect 3 times.
The x values are x = 6, 1.624 and – 0.7899

CHECK:

$$6^{(-0.7899)} = 0.2429$$
$$(-0.7899)^6 = 0.2429$$

We can have: x = 6, y = – 0.7899
and x = – 0.7899, y = 6

If we draw $Y = 8^x$ and $Y = x^8$ we also get graphs which intersect 3 times.
The x values are x = 8, 1.463 and – 0.8101

CHECK:

$$8^{(-0.8101)} = 0.1855$$
$$(-0.8101)^8 = 0.1855$$

We can have: x = 8, y = – 0.8101
and x = – 0.8101, y = 8

If we draw $Y = 10^x$ and $Y = x^{10}$ we also get graphs which intersect 3 times.
The x values are x = 10, 1.371 and – 0.8267

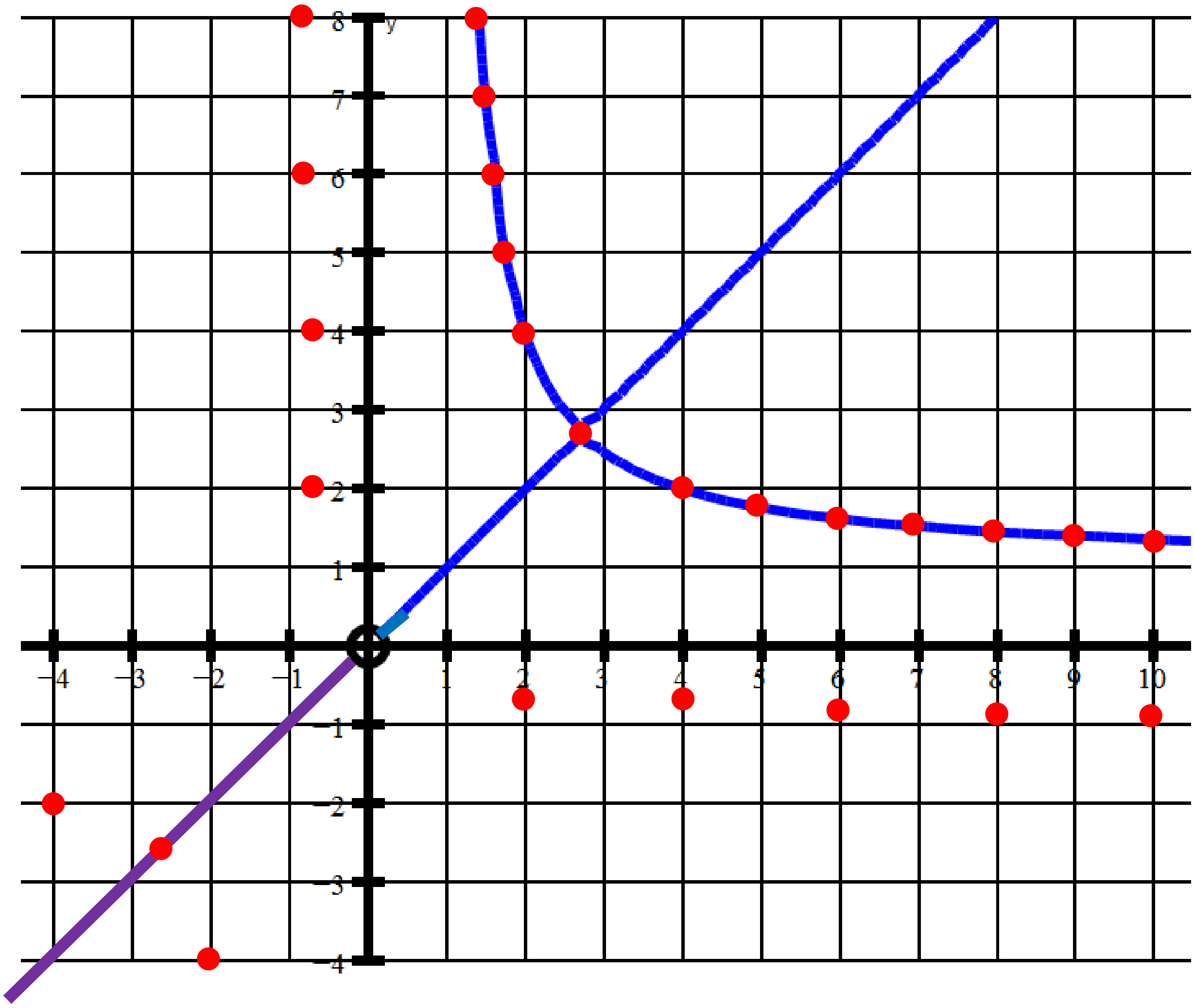
CHECK:

$$10^{(-0.8267)} = 0.1490$$
$$(-0.8267)^{10} = 0.1490$$

We can have: x = 10, y = – 0.8267
and x = – 0.8267, y = 10

These are all the solutions of $y^x = x^y$ I have found so far!

See below:



I have not YET found any MORE!!!

Apart from the infinite complex solutions of the form $x = y = a + ib$ there are no actual "PHANTOM CURVES" because phantom graphs require A COMPLEX PLANE AND A REAL AXIS.

If the equation $y^x = x^y$ had any complex solutions such as $x = a + bi$ and $y = c + id$ then we would need a complex x plane and a complex y plane which would require 4 dimensional space. However, there may be some values a, b, c, d such that $(a + ib)^{(c + id)} = (c + id)^{(a + ib)}$ but I am still looking!

In 2017, I found a NEW set of points…………..

NEW POINTS ON THE GRAPH $y^x = x^y$

Thanks to Marcelo Arruda from BRAZIL for the following ideas.

Using the basic form of De Moivre's theorem: if $z = rcis\theta = (\cos\theta + i\sin\theta)$
then: $z^n = r^n cisn\theta = r^n(\cos n\theta + i\sin n\theta)$, so if z is to be REAL then $\sin n\theta$ must be zero so $\boldsymbol{\theta = n\pi}$ (ie multiples of π rads or 180^0)
Therefore if z is to be real then $z^n = r^n(\cos n\pi + i\sin n\pi)$

Then, considering two **negative** real numbers "***a***" and "***b***", (N.B. the modulus is always positive so the **modulus** of "***a***" is "***– a***") then we can write:

$$z^n = \quad r^n \quad (\cos n\theta + i\sin n\theta)$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$a^b = (-a)^b(\cos b\pi + i\sin b\pi) = \left((-\boldsymbol{a})^{-\boldsymbol{b}}\right)^{-1}(\cos b\pi + i\sin b\pi)$$

$$b^a = (-b)^a(\cos a\pi + i\sin a\pi) = ((-\boldsymbol{b})^{-\boldsymbol{a}})^{-1}(\cos a\pi + i\sin a\pi)$$

Now we will look at the two parts of these expressions and analyse them individually:

$$a^b = ((-a)^{-b})^{-1}(\boldsymbol{\cos b\pi + i\sin b\pi})$$

$$b^a = ((-b)^{-a})^{-1}(\boldsymbol{\cos a\pi + i\sin a\pi})$$

So, if the **positive** numbers $x = -a$ and $y = -b$ satisfy $x^y = y^x$, then the **red** parts of above equations show this result and must be equal to each other.
(ie The equation: $x^y = y^x$ becomes $(-a)^{-b} = (-b)^{-a}$)

The **blue** parts will be equal to each other if:
$\cos \boldsymbol{b}\pi = \cos \boldsymbol{a}\pi$ and $\sin \boldsymbol{b}\pi = \sin \boldsymbol{a}\pi$, which means $\boldsymbol{b\pi = a\pi \pm 2k\pi}$ and therefore $\boldsymbol{b = a \pm 2k}$.(where $\boldsymbol{k}$ is any whole number)

Recall ***a*** and ***b*** are negative so multiplying that last equality by *-1* we get
$\boldsymbol{-b = -a \pm 2k}$ (remember "***– b***" and "***– a***" are positive numbers!)

So, if we can find pairs of **<u>positive numbers</u>** ***x*** and ***y*** which differ by $\boldsymbol{2k}$ and which obey $\boldsymbol{x^y = y^x}$, then their **opposite negative numbers** $-x \; and -y$ will also satisfy the equation $(-x)^{(-y)} = (-y)^{(-x)}$.
The simplest example of this is when ***x*** = 4 and ***y*** = 2. These numbers differ by 2 and they satisfy $4^2 = 2^4$ so this means that the **opposites** ***x*** = – 4 and ***y*** = – 2 will also satisfy the equation: $\boldsymbol{x^y = y^x}$ **because** $\boldsymbol{(-4)^{(-2)} = (-2)^{(-4)}}$

To find such numbers, we let $\boldsymbol{y = x - 2k}$
and solve $\boldsymbol{x^{x-2k}} = (\boldsymbol{x - 2k})^{\boldsymbol{x}}$ --------------------EQU 1
for $k = 1, 2, 3$ and so on.

<u>Examples:</u>
If **k = 1**, Equ. 1 becomes $x^{x-2} = (x-2)^x$, whose solution is $x = 4$.
(Found by drawing the graphs $f(x) = x^{x-2} and\ f(x) = (x-2)^x$ using the AUTOGRAPH program and finding the intersection point.)
This leads to $x = 4\ and\ y = x - 2 = 2$. The positive solutions are **+4** and **+2** and therefore, $\mathbf{-4}$ and $\mathbf{-2}$ will also satisfy $\boldsymbol{x^y = y^x}$
(In each case, the ***x*** and ***y*** values can be swapped to produce $x = -2\,, y = -4$)
(We already knew these solutions.)

Now, let's explore some new solutions using $\boldsymbol{y = x - 2k}$:

If **k = 2** (so the ***x*** and ***y*** differ by 4) then Equ 1 becomes $x^{x-4} = (x-4)^x$ whose solution is $x = 5.6647143$ (from Autograph)
This leads to $y = x - 4 = 1.6647143$,
so $\boldsymbol{x = -5.6647143}$ and $\boldsymbol{y = -1.6647143}$ will be solutions too.
Testing: $(-5.6647143)^{-1.6647143} = 0.0275738 + 0.048443i$
$(-1.6647143)^{-5.6647143} = 0.0275738 + 0.048443i$
(Again we can say $\boldsymbol{x = -1.6647143}$ and $\boldsymbol{y = -5.6647143}$ are solutions)

If **k = 3** (so the ***x*** and ***y*** differ by 6) then Equ 1 becomes $x^{x-6} = (x-6)^x$, whose solution is $x = 7.4941717$
This leads to $y = x - 6 = 1.4941717$,
so $\boldsymbol{x = -7.4941717}$ and $\boldsymbol{y = -1.4941717}$ will be solutions too.
Testing: $(-7.4941717)^{-1.4941717} = -0.000903 + 0.049311i$
$(-1.4941717)^{-7.4941717} = -0.000903 + 0.049311i$
(Again $\boldsymbol{x = -1.4941717}$ and $\boldsymbol{y = -7.4941717}$ are solutions too.)

If **k = 4** (so the ***x*** and ***y*** differ by 8) then Equ 1 becomes $x^{x-8} = (x-8)^x$, whose solution is $x = 9.3944668$
This leads to $y = x - 8 = 1.3944668$,
so $\boldsymbol{x = -9.3944668}$ and $\boldsymbol{y = -1.3944668}$ will be solutions too.
Testing: $(-9.3944668)^{-1.3944668} = -0.014319 + 0.041595i$
$(-1.3944668)^{-9.3944668} = -0.014319 + 0.041595i$
(Again $\boldsymbol{x = -1.3944668}$ and $\boldsymbol{y = -9.3944668}$ are solutions too**.)**

If **k = 5** (so the ***x*** and ***y*** differ by 10) then Equ 1 becomes $x^{x-10} = (x-10)^x$, whose solution is $x = 11.33$
This leads to $y = x - 10 = 1.33$
so $\boldsymbol{x = -11.33}$ and $\boldsymbol{y = -1.33}$ will be solutions too.
Testing: $(-11.33)^{-1.33} = -0.020 + 0.0340i$
$(\ -1.33)^{-11.33} = -0.020 + 0.0340i$
(Again $\boldsymbol{x = -1.33}$ and $\boldsymbol{y = -11.33}$ are solutions too.)

We can continue this as far as we like, but the pattern is better seen graphically.

The **intersection points** (●) of the “*hyperbola-like*” curve in the 1st quadrant, of positive ***x and y*** solutions of $x^y = y^x$, with the lines $\boldsymbol{y = x, y = x \pm 2, y = x \pm 4, y = x \pm 6}$ etc., **are reflected in the line $\boldsymbol{y = -x}$ so that they re-appear in the 3rd quadrant but with the negative versions of the coordinates.**

We already knew the points (–4, –2), (–2.718, –2.718) and (–2, –4).
The solutions to $x^y = y^x$ previously known are **all the points on the purple line $\boldsymbol{y = x}$, all the points on the blue “hyperbola-like curve**” and all the points denoted by red dots (●)
The LIGHT BLUE points (●) are the new ones. (Thanks to Marcelo Arruda from BRAZIL)